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Reductions of integrable lattices

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Abstract
Based on the notion of Darboux–KP chain hierarchy and its invariant submanifolds we construct some class of constraints compatible with integrable lattices. Some simple examples are given.

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1. Introduction

Our concern in this paper is about differential-difference systems (lattices) over a finite number of fields (unknown functions of discrete variable i) which share the property of having an infinite number of conservation laws. In a sense one can interpret these conservation laws as an analog of first integrals for finite-dimensional dynamical systems. It is supposed that each conservation law corresponds to the flow on a suitable infinite-dimensional phase space and these flows are pair-wise commuting. Differential-difference systems of this type are referred to as 'integrable'. For the integrable lattice under consideration corresponding flows governed by some evolutionary equations can be interpreted as generalized symmetries. One says that the given system with these properties admits an integrable hierarchy (see, for example, [10]).

A number of differential-difference systems with the above-mentioned properties which have applications in different areas of natural sciences are known from the literature. Perhaps the simplest and on the other hand the interesting example of an integrable lattice is the well-known Volterra [22] (or Kac–van Moerbeke) equation [5].

$$a_i' = a_i(a_{i-1} - a_{i+1}). \tag{1}$$

It can be considered as a single equation on the function $a = a(i, x) = a_i$ of discrete variable $i \in \mathbf{Z}$ and continuous variable $x \equiv t_1 \in \mathbf{R}$ and is known to be the integrable discretization of the Korteweg–de Vries equation [5]. Due to numerous applications of the Volterra lattice (see, for example, [4, 8, 21]), it can be accepted nowadays as a classical equation of mathematical physics. This equation is known to be integrable by the inverse scattering transform method due to [8].

The Volterra lattice hierarchy can be written as an infinite number of evolutionary equations of the form

$$\partial_s a_i = a_i(\zeta_s(i+1) - \zeta_s(i-1)), \qquad \text{where} \quad \partial_s \equiv \frac{\partial}{\partial t_s}. \tag{2}$$

It turns out that polynomial discrete functions $\zeta_k = \zeta_k[a]$ here are conservation densities for all the flows. One can write the differential-difference conservation laws for (2) as

$$D_{t_s}\zeta_k(i) = J_{s,k}(i+1) - J_{s,k}(i).$$

The discrete functions ζ_k can be calculated by making use of the recursion relation

$$\Delta\zeta_{k+1} = L\zeta_k, \qquad \text{with} \quad \zeta_1 = -a$$

which can be derived from the Lax representation of the Volterra lattice hierarchy (see, for example, [19]). This relation is defined by a pair of discrete operators: forward difference operator $\Delta \equiv \Lambda - 1$ and $L \equiv (a\Lambda + a)\circ\Lambda^{-1} - \Lambda\circ(a\Lambda + a)$ with Λ being a shift operator acting on an arbitrary function of discrete variable as $(\Lambda f)(i) = f(i+1)$.

Formally, one can construct first integrals of (2) as $\zeta_k = \sum_{i\in\mathbf{Z}}\zeta_k(i)$ but, generally speaking, it makes no sense because of the convergence problem. Provided that the periodicity condition $a_{i+N} = a_i$ is imposed which is evidently compatible (since the right-hand side of (2) does not depend explicitly on i) with all flows (2) one is forced to consider the finite-dimensional dynamical system

$$u_k' = u_k(u_{k-1} - u_{k+1}) \qquad \text{where} \quad k \in \mathbf{Z}/N\mathbf{Z}.$$

It is known to be the Liouville-integrable Hamiltonian system.

One knows many examples of reductions for the Volterra lattice defined by some algebraic constraints [4]. As in the case of periodic condition, these constraints lead to Liouville-integrable or Painlevé-type systems. In this paper, as a particular result, we present an infinite number of conditions compatible with the Volterra lattice hierarchy which can be written as the ordinary autonomous Nth-order difference equation

$$a_{i+N} = R(a_i, \ldots, a_{i+N-1}) \tag{3}$$

with the right-hand side R being some rational function of its arguments. As a result, we are led to the system of ordinary differential equations

$$\begin{aligned} u_1' &= u_1(S(u_1, \ldots, u_N) - u_2) \\ u_k' &= u_k(u_{k-1} - u_{k+1}), \qquad k = 2, \ldots, N-1 \\ u_N' &= u_N(u_{N-1} - R(u_1, \ldots, u_N)) \end{aligned} \tag{4}$$

on $u_1 = a_i, \ldots, u_N = a_{i+N-1}$ with some initial value $i = i_0$. The function S here is defined by the inversion formula $a_{i-1} = S(a_i, \ldots, a_{i+N-1})$. Shifting $i \to i+1$ yields the discrete symmetry transformation

$$\bar{u}_k = u_{k+1}, \qquad k = 1, \ldots, N-1, \qquad \bar{u}_N = R(u_1, \ldots, u_N)$$

for (4). It should be noted that some class of reductions generated by a kind of relations (3) was previously presented in [16] and all the examples given there can be derived as particular cases of reductions in this paper.

It is our main goal in this paper to present a unified geometric approach to constructing some class of restrictions compatible with integrable lattices and its hierarchies. We consider a community of integrable lattices which are supposed to be derived as a result of suitable reduction of the so-called Darboux–KP (DKP) chain hierarchy which, in fact, is a chain of KP hierarchy solutions related to each other by the Darboux map [7]. From the geometric point of view it is convenient to present the DKP chain in the form of differential and differential-difference conservation laws. One writes its equations in terms of infinite collections $\{h_k, a_k\}$ of functions of evolution parameters t_s and the discrete variable i. It is relevant in this approach to consider these unknown functions as coefficients of some formal Laurent series.

The DKP chain hierarchy admits an infinite number of invariant submanifolds $\mathcal{S}^n_{l-1}$ and the Volterra equation and many other integrable lattices arise as a result of restriction on the DKP chain on the submanifold constructed as an intersection $\mathcal{S}^n_0 \cap \mathcal{S}^p_{l-1}$ with some n, p and l. In particular $\mathcal{S}^1_0 \cap \mathcal{S}^2_0$ corresponds to the Volterra lattice. It is important that all integrable lattices under consideration in this paper may be written in terms of universal coordinates $\{a_k = a_k(i) : k \geqslant 1\}$ parameterizing the points of some infinite-dimensional phase space $\mathcal{M}$. Restricting the DKP chain only to $\mathcal{S}^n_0$ yields a hierarchy which is more convenient to write in terms of an infinite number of functions $\{a_k\}$. This hierarchy we call the nth discrete KP, since in the case $n = 1$ one has equations of ordinary discrete KP hierarchy. Integrable lattices over a finite number of fields can be considered as a result of restriction of $\mathcal{M}$ on some submanifold $\mathcal{M}_{n,p,l}$ with the help of algebraic equations

$$I_k[a_1, a_2, \ldots] = 0, \qquad \text{with} \quad k \geqslant 1$$

where I_k are some suitable polynomial discrete functions on $\mathcal{M}$.

To make the matter more clear, let us illustrate our general result stated below in theorem 3 by discussing a simple example. Consider the one-field lattice

$$a_i' + a_{i+1}' = (a_i + a_{i+1})(a_{i-1} - a_{i+2})$$

which corresponds to $\mathcal{S}^1_0 \cap \mathcal{S}^3_0$ and requires that a also be a solution to the Volterra equation. It is equivalent to the condition

$$a_i a_{i+2} = a_{i+1} a_{i+3} \tag{5}$$

which, as can be checked by direct calculations, is compatible with the Voltera lattice. At the first sight it seems that the Volterra equation supplemented by constraint (5) corresponds to the triple intersection $\mathcal{S}^1_0 \cap \mathcal{S}^2_0 \cap \mathcal{S}^3_0$, but one can check that it gives only its trivial solution $a = 0$. The point is that there exist invariant conditions weaker than those defining $\mathcal{M}_{1,3,1}$. It turns out that these conditions are written in the form of periodicity relations with some discrete polynomial functions. The submanifold defined by these 'weak' conditions we denote as $\mathcal{N}_{1,3,1}$ and the Volterra equation supplemented by (5) appears as a result of restriction of discrete KP hierarchy on $\mathcal{M}_{1,2,1} \cap \mathcal{N}_{1,3,1}$.

This paper is organized as follows: in section 2, we provide the reader with some basic facts about the DKP chain hierarchy following the lines suggested in [7]. In section 3, we formulate the theorem which provides us with an infinite number of invariant submanifolds for the DKP chain hierarchy. This result is the basis to establish a relationship of many integrable lattices with the KP hierarchy. We provide the reader with some examples of integrable differential-difference systems over a finite number of fields. In section 4 we formulate our main result which allows us to construct a broad class of constraints compatible with integrable lattices and show, in section 5, how this result can be applied on the example of extended Volterra equation [9]

$$a_i' = a_i \left(\sum_{j=1}^{n} a_{i-j} - \sum_{s=1}^{n} a_{i+j} \right) \tag{6}$$

also known as Bogoyavlenskii lattices [3].

2. The DKP chain hierarchy

Let us first give some basics on the DKP chain [7]. This can be defined by two relations

$$\partial_s h(i) = \partial H^{(s)}(i), \tag{7}$$

$$\partial_s a(i) = a(i)(H^{(s)}(i+1) - H^{(s)}(i)) \tag{8}$$

the first of which yields evolution equations of the KP hierarchy in the form of local conservation laws with $h = z + \sum_{k\geqslant 2} h_k z^{-k+1}$ being a generating function for conserved densities of the KP hierarchy. The formal Laurent series

$$H^{(s)} = z^s + \sum_{k\geqslant 1} H_k^s z^{-k},$$

attached to any integer $s \geqslant 2$ is the generating function for suitable fluxes and can uniquely be defined as the projection of z^s on the space

$$\mathcal{H}_+ = \langle 1, h, h^{(2)}, \ldots \rangle$$

spanned by Faà di Bruno iterates $h^{(k)} \equiv (\partial + h)^k(1)$. For instance, one has

$$H^{(1)} = h, \qquad H^{(2)} = h^{(2)} - 2h_2, \qquad H^{(3)} = h^{(3)} - 3h_2 h - 3h_3 - 3h_2', \ldots. \tag{9}$$

Thus the coefficients H_k^s are defined as differential polynomials of $h_2, h_3, \ldots$. The linear relations (9) are invertible and this means that any element of $\mathcal{H}_+$ can be written as a suitable linear combination over $H^{(k)}$.

The representation of the KP hierarchy in the form (7) is equivalent to Sato's formulation of that on the level of Lax equation

$$\partial_s \mathcal{Q} = [(\mathcal{Q}^s)_+, \mathcal{Q}],$$

on the formal pseudodifferential operator $\mathcal{Q} = \partial + \sum_{k\geqslant 2} u_k \partial^{-k+1}$. One has the following:

$$\partial = \mathcal{Q} + \sum_{k\geqslant 2} h_k \mathcal{Q}^{-k+1}$$

and

$$(\mathcal{Q}^s)_+ = \mathcal{Q}^s + \sum_{k\geqslant 2} H_k^s \mathcal{Q}^{-k+1}.$$

One can write, for example, the following:

$$h_2 = -u_2, \qquad h_3 = -u_3, \qquad h_4 = -u_4 - u_2^2, \ldots$$

and

$$H_1^2 = 2h_3 + h_2' = -2u_3 - u_2', \qquad H_2^2 = 2h_4 + h_3' + h_2^2 = -2u_4 - u_3' - u_2^2, \ldots.$$

Relation (8) on the formal Laurent series $a(i) = z + \sum_{k\geqslant 1} a_k(i) z^{-k+1}$ relates two neighbors $h(i)$ and $h(i+1)$ by the Darboux map $h(i) \to h(i+1) = h(i) + a_x(i)/a(i)$ and guarantees compatibility of the latter with KP flows. It is obvious that this equation can be rewritten in the form of differential-difference conservation laws

$$\partial_s \xi(i) = H^{(s)}(i+1) - H^{(s)}(i)$$

with

$$\begin{aligned}\xi = \ln a &= \ln z + \sum_{k\geqslant 1} a_k z^{-k} - \frac{1}{2}\left(\sum_{k\geqslant 1} a_k z^{-k}\right)^2 + \frac{1}{3}\left(\sum_{k\geqslant 1} a_k z^{-k}\right)^3 - \cdots \\ &\equiv \ln z + \sum_{k\geqslant 1} \xi_k z^{-k}.\end{aligned}$$

Thus, we consider the following equations of DKP chain hierarchy:

$$\partial_s h_k(i) = \partial H^s_{k-1}(i), \qquad \partial_s \xi_k(i) = H^s_k(i+1) - H^s_k(i). \tag{10}$$

3. Integrable lattices

3.1. Invariant submanifolds of the DKP chain

As was mentioned in the introduction one knows from the literature many examples of differential-difference systems over a finite number of fields which share the property of having an infinite number of conservation laws and the corresponding one-parametric generalized symmetry groups defined by respective evolutionary equations. There are different methods for constructing integrable lattices and its explicit solutions, such as Lax pairs, recursion operators etc. For details, see, for example, [1–3, 5, 6, 9, 10, 12, 17].

In [14, 15], we have proved that the DKP chain is a convenient and simple notion of showing relationship of integrable lattices with the KP hierarchy. This relationship is very useful because of the remarkable Sato theory which gives description of analytic solutions of the KP hierarchy in terms of an infinite Grassmanian manifold and τ-function [11]. The following two theorems give a framework for constructing integrable lattices whose hierarchies directly relate with the KP hierarchy.

Theorem 1 [15]**.** *The submanifold $\mathcal{S}^n_{l-1}$ defined by the condition*

$$z^{l-n}a^{[n]}(i) \in \mathcal{H}_+(i), \qquad \forall i \in \mathbf{Z} \tag{11}$$

is tangent with respect to the DKP chain flows defined by (10).

Theorem 2 [16]**.** *The chain of inclusions of invariant submanifolds*

$$\mathcal{S}^n_{l-1} \subset \mathcal{S}^{2n}_{2l-1} \subset \mathcal{S}^{3n}_{3l-1} \subset \cdots \subset \mathcal{S}^{kn}_{kl-1} \subset \cdots$$

is valid.

Here, by definition

$$a^{[s]}(i) = \begin{cases} a(i) \times \cdots \times a(i+s-1), & s \geqslant 1 \\ 1, & s = 0 \\ a^{-1}(i-1) \times \cdots \times a^{-1}(i-|s|), & s \leqslant -1 \end{cases}$$

are discrete Faà di Bruno iterates of Laurent series $a(i)$. In what follows, we use the simple obvious identity

$$a^{[r_1+r_2]}(i) = a^{[r_1]}(i)a^{[r_2]}(i+r_1) = a^{[r_2]}(i)a^{[r_1]}(i+r_2), \qquad \forall r_1, r_2 \in \mathbf{Z}. \tag{12}$$

Using the coefficients $a^{[r]}_k$ defined by the relation $a^{[k]} = z^k + \sum_{s\geqslant 1} a^{[k]}_s z^{k-s}$ one recovers from (12)

$$\begin{aligned} a^{[r_1+r_2]}_k(i) &= a^{[r_1]}_k(i) + \sum_{j=1}^{k-1} a^{[r_1]}_j(i)a^{[r_2]}_{k-j}(i+r_1) + a^{[r_2]}_k(i+r_1) \\ &= a^{[r_2]}_k(i) + \sum_{j=1}^{k-1} a^{[r_2]}_j(i)a^{[r_1]}_{k-j}(i+r_2) + a^{[r_1]}_k(i+r_2). \end{aligned} \tag{13}$$

We observe that the condition (11) can be written in the form of the following generating relation:

$$z^{l-n}a^{[n]} = H^{(l)} + \sum_{k=1}^{l} a^{[n]}_k H^{(l-k)}.$$

After a look at the negative powers of z in the latter formula, one gets the explicit form of (11) as follows:

$$G_k^{(n,l)} \equiv a_{k+l}^{[n]} - H_k^l - \sum_{j=1}^{l-1} a_j^{[n]} H_k^{l-j} = 0, \qquad \forall k \geqslant 1.$$

3.2. *Restriction of the DKP chain on* $\mathcal{S}_0^n$

Let us consider the case $l = 1$ corresponding to the invariant submanifold $\mathcal{S}_0^n$. It is defined by conditions

$$G_k^{(n,1)} \equiv a_{k+1}^{[n]} - H_k^1 = 0, \qquad \forall k \geqslant 1.$$

So, on $\mathcal{S}_0^n$ one has $H_k^1 = h_{k+1} = a_{k+1}^{[n]}$. Theorem 2 says that

$$\mathcal{S}_0^n \subset \mathcal{S}_1^{2n} \subset \mathcal{S}_2^{3n} \subset \cdots \subset \mathcal{S}_{k-1}^{kn} \subset \cdots$$

and, hence,

$$G_k^{(n,1)} = 0 \quad \Rightarrow \quad G_k^{(2n,2)} = 0 \quad \Rightarrow \quad G_k^{(3n,3)} = 0 \quad \Rightarrow \quad \cdots$$

Solving successively these relations in favor of H_k^s gives

$$H_k^s = F_k^{(n,s)}[a_1, a_2, \ldots] \equiv a_{k+s}^{[sn]} + \sum_{j=1}^{s-1} q_j^{(n,sn)} a_{k+s-j}^{[(s-j)n]}, \tag{14}$$

where $q_k^{(n,r)} = q_k^{(n,r)}[a_1, a_2, \ldots]$ are polynomial discrete functions defined by the relation

$$z^r = a^{[r]} + \sum_{k\geqslant 1} q_k^{(n,r)} z^{k(n-1)} a^{[r-kn]} \tag{15}$$

or, more exactly,

$$a_k^{[r]} + \sum_{j=1}^{k-1} a^{[r-jn]} q_j^{(n,r)} + q_k^{(n,r)} = 0, \qquad \forall k \geqslant 1. \tag{16}$$

Below the first few $q_k^{(n,r)}$ are written as

$$q_1^{(n,r)} = -a_1^{[r]}, \qquad q_2^{(n,r)} = -a_2^{[r]} + a_1^{[r]} a_1^{[r-n]},$$
$$q_3^{(n,r)} = -a_3^{[r]} + a_1^{[r]} a_2^{[r-n]} + a_1^{[r-2n]} a_2^{[r]} - a_1^{[r]} a_1^{[r-n]} a_1^{[r-2n]}.$$

It can be shown that the functions $q_k^{(n,r)}$ are related to each other by the relation

$$\begin{aligned} q_1^{(n,r_1+r_2)}(i) &= q_k^{(n,r_1)}(i) + \sum_{j=1}^{k-1} q_j^{(n,r_1)}(i) q_{k-j}^{(n,r_2)}(i + r_1 - jn) + q_k^{(n,r_2)}(i + r_1) \\ &= q_k^{(n,r_2)}(i) + \sum_{j=1}^{k-1} q_j^{(n,r_2)}(i) q_{k-j}^{(n,r_1)}(i + r_2 - jn) + q_k^{(n,r_1)}(i + r_2). \end{aligned} \tag{17}$$

In what follows, there is a need for the more general than (16) relation

$$a_k^{[r]}(i) + \sum_{j=1}^{k-1} a_{k-j}^{[r-jn]}(i) q_j^{(n,r-p)}(i + p) + q_k^{(n,r-p)}(i + p) = a_k^{[p]}(i) \tag{18}$$

with any integers r and p. The latter can easily be obtained as follows. Taking into account (12), we obtain

$$\begin{aligned} z^{r-p} &= a^{[r-p]}(i+p) + \sum_{k\geqslant 1} q_k^{(n,r-p)}(i+p) z^{k(n-1)} a^{[r-p-kn]}(i+p) \\ &= a^{[-p]}(i+p)\left(a^{[r]}(i) + \sum_{k\geqslant 1} q_k^{(n,r-p)}(i+p) z^{k(n-1)} a^{[r-kn]}(i)\right) \end{aligned}$$

and

$$z^{r-p} a^{[p]}(i) = a^{[r]}(i) + \sum_{k\geqslant 1} q_k^{(n,r-p)}(i+p) z^{k(n-1)} a^{[r-kn]}(i).$$

Then writing explicitly the latter relation we get (18). Solving (18) in favor of $q_k^{(n,r-p)}(i+p)$ yields

$$a_k^{[p]}(i) + \sum_{j=1}^{k-1} q_j^{(n,r-(k-j)n)}(i) a_{k-j}^{[p]}(i) + q_k^{(n,r)}(i) = q_k^{(n,r-p)}(i+p). \tag{19}$$

Observe that relations (14) are coded in

$$H^{(s)} = z^{s(1-n)} a^{[sn]} + \sum_{j=1}^{s} z^{(s-j)(1-n)} q_j^{(n,sn)} a^{[(s-j)n]}.$$

This means that when restricting to $\mathcal{S}_0^n$, one has

$$\mathcal{H}_+ = \langle 1, z^{1-n} a^{[n]}, z^{2(1-n)} a^{[2n]}, \ldots \rangle. \tag{20}$$

We see that on the $\mathcal{S}_0^n$ DKP chain equations can be written in the form of differential-difference conservation laws

$$\partial_s \xi_k(i) = F_k^{(n,s)}(i+1) - F_k^{(n,s)}(i). \tag{21}$$

More exactly, equations (21) appear as a result of projection of restricted DKP chain flows on the space $\mathcal{M}$ whose points are defined by an infinite number of functions $\{a_k = a_k(i)\}$. As was shown in [7], with $n = 1$, the evolution equations (21) are equivalent to the discrete KP hierarchy (dKP) [18]. We can show that given any solution of DKP chain hierarchy restricted to $\mathcal{S}_0^n$, the noninvertible map $g_n : a(i) \to z^{1-n} a^{[n]}(ni)$ gives the solution of the dKP one. More generally, it is known that $g_k\left(\mathcal{S}_{l-1}^{kn}\right) \subset \mathcal{S}_{l-1}^n$ [15].

We refer, for simplicity, to equations (21), with some fixed n as dKP hierarchy. Since all the nth dKP hierarchy flows 'live' on the same phase space $\mathcal{M}$ it is natural to call (21) (with any n) the extended dKP hierarchy [13].

It is useful for our aims to write equations

$$\begin{aligned} D_{t_s} q_k^{(n,r)}(i) &= q_{k+s}^{(n,r)}(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i) q_{k+s-j}^{(n,r)}(i+(s-j)n) \\ &\quad - q_{k+s}^{(n,r)}(i) - \sum_{j=1}^{s} q_j^{(n,sn)}(i+r-(k+s-j)n) q_{k+s-j}^{(n,r)}(i) \end{aligned} \tag{22}$$

which are fulfilled by virtue of (21). These equations can be found by making use of the suitable differential-difference Lax equation (see, for example, [15]).

3.3. Restriction of the DKP chain on $\mathcal{S}_0^n \cap \mathcal{S}_{l-1}^p$

Let us now consider nontrivial intersections $\mathcal{S}_0^n \cap \mathcal{S}_{l-1}^p$. The word 'nontrivial' means that $ln - p \neq 0$ is to be supposed. According to theorem 1, the restriction of the DKP chain on $\mathcal{S}_0^n \cap \mathcal{S}_{l-1}^p$ is defined by the generating relation

$$z^{l-p}a^{[p]} - F^{(n,l)} - \sum_{j=1}^{l} a_j^{[p]} F^{(n,l-j)} = \sum_{k\geqslant 1} J_k^l z^{-k} = 0$$

with the Laurent series

$$F^{(n,s)} \equiv z^s + \sum_{k\geqslant 1} F_k^{(n,s)} z^{-k}.$$

Thus, the restriction of DKP chain flows on $\mathcal{S}_0^n \cap \mathcal{S}_{l-1}^p$ is given by the following polynomial conditions:

$$J_k^l \equiv a_{k+l}^{[p]} - F_k^{(n,l)} - \sum_{j=1}^{l-1} a_j^{[p]} F_k^{(n,l-j)} = 0, \qquad \forall k \geqslant 1 \tag{23}$$

which, in fact, define some submanifold $\mathcal{M}_{n,p,l} \subset \mathcal{M}$ invariant with respect to flows of the nth discrete KP hierarchy (21). Making use of relation (19) we get the following,

$$F_k^{(n,l)}(i) + \sum_{j=1}^{l-1} a_j^{[p]}(i) F_k^{(n,l-j)}(i) = a_{k+l}^{[ln]}(i) + \sum_{j=1}^{l-1} q_j^{(n,ln-p)}(i+p) a_{k+l-j}^{[(l-j)n]}(i),$$

and, hence,

$$J_k^l = a_{k+l}^{[p]}(i) - a_{k+l}^{[ln]}(i) - \sum_{j=1}^{l-1} q_j^{(n,ln-p)}(i+p) a_{k+l-j}^{[(l-j)n]}(i).$$

Replacing in (18) $k \to k+l$ and setting $r = ln$ we get

$$J_k^l = Q_k^l + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]} Q_{k-j}^l, \qquad k \geqslant 1, \tag{24}$$

with $Q_k^l(i) \equiv q_{k+l}^{(n,ln-p)}(i+p)$. Solving these relations in favor of Q_k^l yields

$$Q_k^l = J_k^l + \sum_{j=1}^{k-1} q_j^{(n,-(k-j)n)} J_{k-j}^l, \qquad k \geqslant 1. \tag{25}$$

From (22) we have

$$\begin{aligned} D_{t_s} Q_k^l(i) = {} & Q_{s+k}^l(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i+p) Q_{s+k-j}^l(i+(s-j)n) \\ & - Q_{s+k}^l(i) - \sum_{j=1}^{s} q_j^{(n,sn)}(i-(s+k-j)n) Q_{s+k-j}^l(i). \end{aligned}$$

It is worthwhile to notice that the coefficients of this equation do not depend on l. The same is true for the coefficients of transformation (25).

3.4. Examples of integrable lattices

Let us give some examples of differential-difference equations below which appear as a result of restriction of the nth dKP hierarchy on $\mathcal{M}_{n,p,l}$.

(i) *One-field lattices*. The submanifold $\mathcal{M}_{n,p,1}$ is defined by an infinite set of conditions $J_k^1 = a_k^{[p]} - a_k^{[n]} = 0, \forall k \geqslant 2$. Without loss of generality one can suppose that $p \geqslant 1$ and $n < p$ and $n \neq 0$. From (13) one has

$$a_2^{[p]}(i) = a_2^{[n]}(i) + a_1^{[n]}(i)a_1^{[p-n]}(i+n) + a_2^{[p-n]}(i+n)$$
$$= a_2^{[n]}(i+p-n) + a_1^{[n]}(i+p-n)a_1^{[p-n]}(i) + a_2^{[p-n]}(i)$$

and

$$a_2^{[p-n]}(i) = -a_1^{[n]}(i-n)a_1^{[p-n]}(i)$$

and

$$a_2^{[n]}(i+p-n) - a_2^{[n]}(i) = a_1^{[p-n]}(i)\big(a_1^{[n]}(i-n) - a_1^{[n]}(i+p-n)\big).$$

From the latter one has the following differential-difference equation[1]:

$$\partial a_1^{[p-n]}(i) = a_i' + \cdots + a_{i+p-n-1}' = a_2^{[n]}(i+p-n) - a_2^{[n]}(i)$$
$$= a_1^{[p-n]}(i)\big(a_1^{[n]}(i-n) - a_1^{[n]}(i+p-n)\big). \quad (26)$$

One needs to consider two cases. Let $n \geqslant 1$ and $p \geqslant n+1$. Then (26) is specified as

$$\sum_{s=1}^{p-n} a_{i+s-1}' = \sum_{s=1}^{p-n} a_{i+s-1}\left(\sum_{s=1}^{n} a_{i-s} - \sum_{s=p-n}^{p-1} a_{i+s}\right). \quad (27)$$

The important case to consider is $n \geqslant 1$ and $p = n+1$ which corresponds to the Bogoyavlenskii lattice (6). Let $n \leqslant -1$ and $p \geqslant 1$. In this case (26) becomes

$$\sum_{s=1}^{p+|n|} a_{i+s-1}' = \sum_{s=1}^{p+|n|} a_{i+s-1}\left(\sum_{s=p+1}^{p+|n|} a_{i+s-1} - \sum_{s=1}^{|n|} a_{i+s-1}\right). \quad (28)$$

It should be noted that two pairs of integers (n, p) and $(-p, |n|)$ correspond to the same equation of the form (28).

(ii) *Toda lattice*. When restricting the dKP hierarchy to $\mathcal{M}_{1,1,2}$, one requires

$$J_k^2 = a_{k+2}(i) - a_{k+2}^{[2]}(i) - q_1^{(1,1)}(i+1)a_{k+1}(i) = 0.$$

The latter is solved by

$$a_{k+2}(i) = -\sum_{j=1}^{k} a_j(i-1)a_{k-j+2}(i).$$

Equations of the first flow of the dKP hierarchy in this case are reduced to a pair of evolution equations

$$a_1'(i) = a_2(i+1) - a_2(i), \qquad a_2'(i) = a_2(i)(a_1(i-1) - a_2(i))$$

which are equivalent to the Toda lattice in its exponential form [17].

[1] Here and in what follows $a_i \equiv a_1(i)$.

(iii) *Belov–Chaltikian lattice*. Restricting the first flow of the dKP hierarchy to $\mathcal{M}_{1,3,2}$ yields the two-field system

$$a_1'(i) = a_2(i+1) - a_2(i),$$
$$a_2'(i) = a_1(i)a_2(i-1) - a_1(i-1)a_2(i+1) + a_2(i)\{a_1(i-2) + a_1(i-1) - a_1(i) - a_1(i+1)\} + a_1(i-1)a_1(i)(a_1(i-2) - a_1(i+1))$$

which, in turn, via the invertible ansatz

$$L_i = -a_1(i), \qquad W_i = a_2(i+1) + a_1(i)a_1(i+1)$$

can be transformed into the Belov–Chaltikian lattice [1]

$$L_i' = W_{i-1} - W_i + L_i(L_{i+1} - L_{i-1}), \qquad W_i' = W_i(L_{i+2} - L_{i-1}).$$

(iv) *Shabat dressing lattice*. Consider $\mathcal{M}_{1,0,2}$ which is defined by conditions

$$J_k^2 = -a_{k+2}^{[2]} - q_1^{(1,2)}a_{k+1} = 0. \tag{29}$$

One can check that by virtue of (29) with $k = 1$,

$$D_x q_2^{(1,2)}(i) = D_x\left(-a_2(i) - a_2(i+1) + a_i^2\right) = 0.$$

So, we can write $a_2(i) + a_2(i+1) = a_i^2 - \mu_i$, where μ_i do not depend on x and

$$a_i' + a_{i+1}' = a_2(i+2) - a_2(i) = a_{i+1}^2 - a_i^2 - \mu_{i+1} + \mu_i.$$

The latter is nothing but the Shabat dressing lattice [12].

4. Reductions of the nth dKP hierarchy

The main goal of this section is to show some class of restrictions compatible with the nth dKP hierarchy (21). The corresponding constraints are supposed to be written in the form of periodicity conditions

$$I_k^l(i+n) = I_k^l(i), \qquad \forall k \geqslant 1 \tag{30}$$

with a suitable infinite collection of polynomial discrete functions $\left\{I_k^l = I_k^l : k \geqslant 1\right\}$. We are looking for these functions through invertible relations[2]

$$Q_k = I_k + \sum_{j=1}^{k-1} \zeta_{k-1,j} I_{k-j}, \qquad k \geqslant 1 \tag{31}$$

with some unknown coefficients $\zeta_{k,j}$. We have

$$D_{t_s} I_1(i) = D_{t_s} Q_1(i) = Q_{s+1}(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i+p) Q_{s-j+1}(i+(s-j)n) - Q_{s+1}(i) - \sum_{k=1}^{s} q_j^{(n,sn)}(i-(s-j+1)n) Q_{s-j+1}(i). \tag{32}$$

Substituting (31) into right-hand side of (32) we require that it be identically zero provided that the conditions (30) are imposed. This leads to the relations

$$\zeta_{s,m}(i+sn) + \sum_{j=1}^{m-1} q_j^{(n,sn)}(i+p)\zeta_{s-j,m-j}(i+(s-j)n) + q_m^{(n,sn)}(i+p) = \zeta_{s,m}(i) + \sum_{j=1}^{m-1} q_j^{(n,sn)}(i-(s-j+1)n)\zeta_{s-j,m-j}(i) + q_m^{(n,sn)}(i-(s-m+1)n)$$

[2] We use sometimes simplified notations $I_k \equiv I_k^l$, $Q_k \equiv Q_k^l$, etc.

with $s \geqslant 1$ and $m = 1, \ldots, s$. Taking into account (17), one can easily check that the solution of these equations is given by

$$\zeta_{s,m}(i) = q_m^{(n,-p-(s-m+1)n)}(i+p). \tag{33}$$

Thus, we have the following: if conditions (30) for I_k defined by the relations

$$Q_k(i) = I_k(i) + \sum_{j=1}^{k-1} q_j^{(n,-p-(k-j)n)}(i+p) I_{k-j}(i), \qquad k \geqslant 1 \tag{34}$$

are valid then $D_{t_s} I_1 = 0, \forall s \geqslant 1$. Solving (34) in favor of I_k yields

$$I_k(i) = Q_k(i) + \sum_{j=1}^{k-1} a_j^{[-p-(k-j)n)]}(i+p) Q_{k-j}(i), \qquad k \geqslant 1. \tag{35}$$

We can prove, by induction, that under (30) the quantities I_k do not depend on evolution parameters t_s for all $k \geqslant 1$. We have

$$\begin{aligned}
D_{t_s} I_{k+1}(i) + \sum_{j=1}^{k} D_{t_s}\zeta_{k,j}(i) I_{k-j+1}(i) + \sum_{j=1}^{k} \zeta_{k,j}(i) D_{t_s} I_{k-j+1}(i) &= D_{t_s} Q_{k+1}(i) \\
&= Q_{s+k+1}(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i+p) Q_{s-j+k+1}(i+(s-j)n) \\
&\quad - Q_{s+k+1}(i) - \sum_{j=1}^{s} q_j^{(n,sn)}(i-(s-j+k+1)n) Q_{s-j+k+1}(i).
\end{aligned}$$

Let us suppose that we have already proved that $D_{t_s} I_j$ for $j = 1, \ldots, k$ by virtue of (30) with I_k are given by (34). Then $D_{t_s} I_{k+1} = 0$ under (30) if the relations

$$\begin{aligned}
\zeta_{s+k,m}(i+sn) &+ \sum_{j=1}^{m-1} q_j^{(n,sn)}(i+p)\zeta_{s+k-j,m-j}(i+(s-j)n) + q_m^{(n,sn)}(i+p) \\
&= \zeta_{s+k,m}(i) + \sum_{j=1}^{m-1} q_j^{(n,sn)}(i-(s+k-j+1)n)\zeta_{s+k-j,m-j}(i) \\
&\quad + q_m^{(n,sn)}(i-(s+k-m+1)n)
\end{aligned}$$

with $m = 1, \ldots, s$ and

$$\begin{aligned}
\zeta_{s+k,s+m}(i+sn) &+ \sum_{j=1}^{s} q_j^{(n,sn)}(i+p)\zeta_{s+k-j,s+m-j}(i+(s-j)n) \\
&= \zeta_{s+k,s+m}(i) + \sum_{j=1}^{s} q_j^{(n,sn)}(i+(s+k-j+1)n))\zeta_{s+k-j,s+m-j}(i) + D_{t_s}\zeta_{k,m}(i)
\end{aligned}$$

with $m = 1, \ldots, k$ are valid. Again we can check that these relations are solved by (33). As an obvious consequence of the above calculations we obtain the following theorem:

Theorem 3. *The periodicity conditions (30) with I_k^l given by (35) are compatible with the nth discrete KP hierarchy.*

Let us denote the submanifold of $\mathcal{M}$ defined by conditions of periodicity (30) as $\mathcal{N}_{n,p,l}$. An infinite set of constraints $I_k^l = 0$ defining $\mathcal{M}_{n,p,l}$ gives a particular solution of (30) and, hence, we have $\mathcal{M}_{n,p,l} \subset \mathcal{N}_{n,p,l}$.

It is useful to establish a relationship between J_k^l and I_k^l. Making use of (18), (24) and (34), we get the relation

$$J_k = I_k + \sum_{j=1}^{k-1} a_j^{[p]} I_{k-j}$$

and its inverse

$$I_k(i) = J_k(i) + \sum_{j=1}^{k-1} a_j^{[-p]}(i+p) J_{k-j}(i).$$

5. Reductions of the Bogoyavlenskii lattice

The goal of this section is to show how theorem 3 can be applied for constructing some class of constraints compatible with the Bogoyavlenskii lattice (6). More exactly, we would like to show a class of restrictions which correspond to intersection $\mathcal{M}_{n,n+1,1} \cap \mathcal{N}_{n,p,1}$ which is evidently equivalent to $\mathcal{M}_{n+1,n,1} \cap \mathcal{N}_{n+1,p,1}$. The submanifold $\mathcal{M}_{n,n+1,1}$ is defined by equations $a_k^{[n]} = a_k^{[n+1]}$ with $k \geqslant 2$ which are uniquely solved as $a_k = P_k^{[n]}[a]$ with some discrete polynomials $P_k^{[n]}$. Next we require that the periodicity conditions $I_k^1(i+n) = I_k^1(i)$ and $I_k^1(i+n+1) = I_k^1(i)$ must be valid simultaneously. This is equivalent to $I_k^1(i+1) = I_k^1(i)$. It is natural to consider three different cases:

(i) Let $p \geqslant n+2$. Keeping in mind the condition $a_2^{[n]} = a_2^{[n+1]}$, one calculates to obtain

$$\begin{aligned} I_1^1(i) = J_1^1(i) &= a_2^{[p]}(i) - a_2^{[n]}(i) \\ &= a_i(a_{i+n+1} + \cdots + a_{i+p-1}) \\ &\quad + a_{i+1}(a_{i+n+2} + \cdots + a_{i+p-1}) + \cdots + a_{i+p-n-2}a_{i+p-1}. \end{aligned}$$

The condition $I_1^1(i+1) = I_1^1(i)$ can be written as

$$a_{i+p} = a_i \frac{Q_{i+n}}{Q_i}, \qquad \text{with} \quad Q_i \equiv \sum_{j=1}^{p-n-1} a_{i+j}. \tag{36}$$

It is natural to suppose that this difference equation has a number of integrals enough for its integrability [20]. If so, then this collection of integrals is provided by $\left\{I_k^1\right\}$ but one must calculate its explicit form $I_k^1 = I_k^1[a]$ in each case. Nevertheless, we can write an explicit form of two integrals for (36):

$$K_i = \frac{\prod_{s=1}^{p} a_{i+s-1}}{\prod_{j=1}^{n} Q_{i+j-1}} \qquad \text{and} \qquad P_i = \frac{\prod_{j=1}^{n+1}(a_{i+j-1} + Q_{i+j-1})}{\prod_{j=1}^{n} Q_{i+j-1}},$$

which directly follows from the form of this equation. The second integral is derived here as follows. As a consequence of (36), we can write

$$a_{i+p}(a_i + Q_i) = a_i(a_{i+n+1} + Q_{i+n+1})$$

and

$$Q_{i+n}(a_i + Q_i) = Q_i(a_{i+n+1} + Q_{i+n+1}).$$

It is obvious that the latter relation can be written as $P_i = P_{i+1}$.

(ii) Let $p \leqslant -1$. We calculate to write

$I_1^1(i) = a_{i-1}(a_{i-|p|} + \cdots + a_{i+n-1}) + a_{i-2}(a_{i-|p|} + \cdots + a_{i+n-2}) + \cdots + a_{i-|p|-n}a_{i-|p|}$.

The corresponding constraint can be written as

$$a_{i+|p|+n}Q_{i+n} = a_{i+n}Q_{i-1}, \qquad \text{with} \quad Q_i \equiv \sum_{j=1}^{|p|+n} a_{i+j}.$$

Solving this relation in favor of $a_{i+|p|+2n}$ yields

$$a_{i+2n+|p|} = \frac{a_{i+n}}{a_{i+|p|+n}}Q_{i-1} - \sum_{j=n+1}^{2n+|p|-1} a_{i+j}. \tag{37}$$

In this case we are also able to write two integrals for (37) in its explicit form

$$K_i = \prod_{j=1}^{|p|} a_{i+j-1} \prod_{j=1}^{n+1} Q_{i-j} \qquad \text{and} \qquad P_i = \frac{\prod_{j=1}^{n}(a_{i-j} + Q_{i-j})}{\prod_{j=1}^{n+1} Q_{i-j}}.$$

(iii) Let $p = 1, \ldots, n-1$. Then

$I_1^1(i) = a_{i-1}(a_{i+p} + \cdots + a_{i+n-1}) + a_{i-2}(a_{i+p} + \cdots + a_{i+n-2}) + \cdots + a_{i+p-n}a_{i+p}$.

We can write the corresponding constraint in the form

$$a_{i+n-p}Q_{i+n} = a_{i+n}Q_{i-1}, \qquad \text{with} \quad Q_i \equiv \sum_{j=1}^{n-p} a_{i+j}$$

and

$$a_{i+2n-p} = \frac{a_{i+n}}{a_{i+n-p}}Q_{i-1} - \sum_{j=n+1}^{2n-p-1} a_{i+j}.$$

The two integrals K and P are

$$K_i = \frac{\prod_{j=1}^{p} a_{i-j}}{\prod_{j=1}^{n+1} Q_{i-j}} \qquad \text{and} \qquad P_i = \frac{\prod_{j=1}^{n}(a_{i-j} + Q_{i-j})}{\prod_{j=1}^{n+1} Q_{i-s}}.$$

One sees that in each of the three cases the corresponding constraints are written in the form of the ordinary difference equation $a_{i+N} = R(a_i, \ldots, a_{i+N-1})$ with $N = p$, $N = 2n+|p|$ and $N = 2n - p$, respectively.

Finally, let us show a simple example corresponding to $p = 4$ and $n = 1$, that is, when the Volterra lattice (1) is constrained by a condition given by the fourth-order difference equation

$$a_{i+4} = a_i\frac{a_{i+2} + a_{i+3}}{a_{i+1} + a_{i+2}}$$

with the three integrals

$$K_i = \frac{a_i a_{i+1} a_{i+2} a_{i+3}}{a_{i+1} + a_{i+2}}, \qquad P_i = \frac{(a_i + a_{i+1} + a_{i+2})(a_{i+1} + a_{i+2} + a_{i+3})}{a_{i+1} + a_{i+2}},$$
$$J_i = a_i(a_{i+2} + a_{i+3}) + a_{i+1}a_{i+3}.$$

Observe that in this case $I_1^1 = J$ and $I_2^1 = -K - JP$. Identify now $a_i = u_1, a_{i+1} = u_2$, $a_{i+2} = u_3, a_{i+3} = u_4$. The attached system of ordinary differential equations looks as follows:

$$u_1' = u_1\left(u_4\frac{u_1 + u_2}{u_2 + u_3} - u_2\right), \qquad u_2' = u_2(u_1 - u_3), \qquad u_3' = u_3(u_2 - u_4),$$

$$u_4' = u_4\left(u_3 - u_1\frac{u_3+u_4}{u_2+u_3}\right).$$

This system has three first integrals

$$K = \frac{u_1u_2u_3u_4}{u_2+u_3}, \qquad P = \frac{(u_1+u_2+u_3)(u_2+u_3+u_4)}{u_2+u_3},$$
$$J = u_1(u_3+u_4) + u_2u_4.$$

Shifting $i \to i+1$ gives a discrete symmetry transformation

$$\bar{u}_k = u_{k+1}, \qquad k = 1, 2, 3, \qquad \bar{u}_4 = u_1\frac{u_3+u_4}{u_2+u_3}. \tag{38}$$

By direct calculations one can check that integrals K, P and J are invariant under (38).

6. Conclusion

We have presented a scheme for constructing a broad class of constraints compatible with integrable lattices which can be derived as reductions of the DKP chain. For a particular case of the Bogoyavlenskii lattice we showed some simple examples in their explicit form. In principle, the solution of the lattice under consideration constrained by some conditions is constructed as follows. One solves the attached system of ordinary differential equations with some initial conditions $\left(u_1^0, \ldots, u_N^0\right) \in \mathbf{R}^N$ to obtain

$$a(i,x) = u_1(x), \ldots, a(i+N-1,x) = u_N(x)$$

for some initial value, $i = i_0$. To find $a(i,x)$ for the remaining values of the discrete variable i one needs to use the discrete symmetry transformation.

In this connection, it is important to determine the structure of solutions for attached systems of ordinary differential equations. We suppose that all these systems are integrable in the Liouville sense with first integrals invariant with respect to the discrete symmetry transformation generated by shifting $i \to i+1$.

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